Arithmetic Progressions

An arithmetic progression is the sum of a set of numbers that are an equal distance apart. For example, the sum

$$3+5+7+9+11+13+15$$

is an arithmetic progression (or AP for short). The 'equal distance' is 2 and the lowest number in the set is 3. Also, the number of numbers in the sequence is 7.

What we wish to do here is develop a formula to give the overall sum, which in the above case is 63.

First, let the sum be S, then we write the equation for S in the following two ways:

$$\begin{array}{ccccccccccccccc} S & = & 3 & + & 5 & + & 7 & + & 9 & + & 11 & + & 13 & + & 15 \\ S & = & 15 & + & 13 & + & 11 & + & 9 & + & 7 & + & 5 & + & 3 \end{array}$$

Now, notice the way I've laid these sums out – the first case is set out with the lowest-valued term first and the second starts with the highest-valued number. Also, the extra spaces between the numbers show that if we add both of the first terms together (the numbers in red), we get $3+15=18$, add both of the second terms (in green) together, we get $5+13=18$, the next pair gives $7+11=18$ and so on. Each pair of numbers totals to 18.

So, adding the two equations together, the left-hand-sides give

$$S+S=2S$$

and the right-hand-sides give

$$\begin{aligned} &3+15+5+13+7+11+9+9+11+7+13+5+15+3 \\ &=18+18+18+18+18+18+18 \\ &=7\times 18 \end{aligned}$$

So, we have

$$2\times S=7\times 18=126$$

or

$$S=\frac{126}{2}=63$$

which is the correct answer!

Well if should be possible to see a formula here, if you look hard enough. Let's look again at the equation

$$2\times S=7\times 18$$

Where does the 18 come from? Well, it's the sum for each pair of terms in the two equations we wrote down at the beginning.

$$
\begin{array}{ccccccccccccccc}
S & = & 3 & + & 5 & + & 7 & + & 9 & + & 11 & + & 13 & + & 15 \\
S & = & 15 & + & 13 & + & 11 & + & 9 & + & 7 & + & 5 & + & 3
\end{array}
$$

In particular, it is the sum of the first and last terms of either equation – the sum of the lowest- and highest-valued terms.

What about the 7? Well this is the number of numbers in the sequence that makes up the sum.

Now, we have to use some algebra. Let a denote the first term of the sequence of numbers and d be the difference between consecutive numbers. Also, let n be the number of numbers in the sequence. Then the last number of the sequence is given by

$$a+(n-1)\times d.$$

So, the sum of the first and last terms (18 in our example) is just

$$a+a+(n-1)\times d = 2\times a+(n-1)\times d$$

and the number of terms (7 in our example) is just n.

So, where our example involved the equation $2\times S = 7\times 18$, the general case will be

$$2\times S = n\times\left(2\times a+(n-1)\times d\right)$$

Dividing throughout by 2, we get the final formula, which is

$$S = a\times n+\frac{d\times n\times(n-1)}{2}.$$

Let's try another example.

$$S = 13+20+27+34+41+48+55+62+69$$

Here, the lowest term of the sequence is 13, the difference between consecutive numbers in the sequence is 7, and the number of terms is 9.

So we must set $a=13$, $d=7$ and $n=9$ and plug these values into the final formula above as follows:

$$S = 13\times 9+\frac{7\times 9\times(9-1)}{2} = 13\times 9+\frac{7\times 9\times 8}{2} = 117+252 = 369,$$

which of course is the right answer. Try a few examples for yourself!