Differentiation from first principles

First of all, 'from first principles' just means 'from basics'. In other words, let's start from scratch and try to differentiate something.

We know what we have to achieve, because differentiation has a definition that we know; for a given function $y = f(x)$, the derivative, $f'(x)$ is defined as

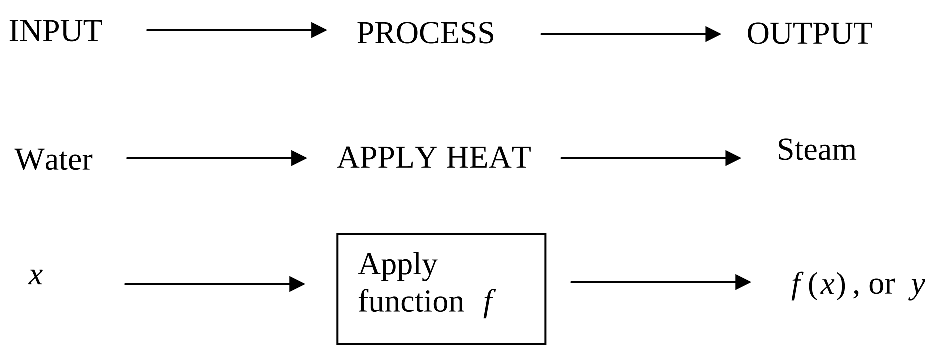
$$f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}.$$

The only thing we need now is an actual function (i.e. a specific $f(x)$) to differentiate – so let's choose one for an example ...

$$y = f(x) = x^2.$$

Let us first make a few remarks about this equation. The variable x is called an input variable, because the function f is applied to it. The variable y is the output variable because it is the result from applying f to x.

For illustration, imagine a simple bunsen burner experiment, the aim of which is to make steam from water. The water is placed in a test tube and so is like an input to the process, heat is applied (the process) and steam is given off as an output. In this experiment, the input is subjected to a process and yields an output. This is shown with its analogy to functions and variables in the diagram below:



So, the x-value is 'fed into' the function and out pops a $f(x)$ value as an output. Notice that the choice of x-value is not important and can be quite arbitrary, whereas the value of y depends on the x-value. For this reason, the input variable is often referred to as the 'independent variable' and the output variable is often known as the 'dependent variable'.

So, back to the exam question, which reads something like ...

Q: Differentiate the function $y = x^2$ using first principles
What is the derivative at $x = 2$ and at $x = -5$?

We have been instructed to go back to basics in this question and not to use any 'previously-understood' facts or special techniques that relate to differentiation.

Well in the first place, we must be able to express what $f(x + \delta x)$ is for our particular function $f(x) = x^2$. Consider what $f(x) = x^2$ means. It means that for any input value, x, to obtain $f(x)$, we square it.

We could have written the equation as $f(y) = y^2$, or as $f(z) = z^2$ - it's just that in maths we *tend* to use x for input variables and y for output variables. What we are being asked to do in obtaining $f(x + \delta x)$ is to use $x + \delta x$ as our independent variable and to square it. So we can write

$$f(x + \delta x) = (x + \delta x)^2$$

in this case. Now all that remains is to work out what $(x + \delta x)^2$ actually is. A common mistake amongst students is to say that $(x + \delta x)^2 = x^2 + \delta x^2$, which is INCORRECT!

When we multiply brackets out, we have to take great care, so first of all, we note that

$$(x + \delta x)^2 = (x + \delta x) \cdot (x + \delta x)$$

then, expanding out the first bracket, we have

$$(x + \delta x) \cdot (x + \delta x) = x \cdot (x + \delta x) + \delta x \cdot (x + \delta x)$$

and finally ...

$$x \cdot (x + \delta x) + \delta x \cdot (x + \delta x) = x^2 + x \cdot \delta x + \delta x \cdot x + \delta x^2 = x^2 + 2x \cdot \delta x + \delta x^2$$

So, recalling that what we are ultimately trying to find is

$$f'(x) = \lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x}$$

we now know that $f(x + \delta x) = x^2 + 2x \cdot \delta x + \delta x^2$, so the fraction within the limit can be written as

$$\frac{f(x + \delta x) - f(x)}{\delta x} = \frac{x^2 + 2x \cdot \delta x + \delta x^2 - x^2}{\delta x} = \frac{2x \cdot \delta x + \delta x^2}{\delta x}$$

Now we can divide the top and bottom parts of this fraction by δx (there is a common factor) to get

$$\frac{f(x+\delta x)-f(x)}{\delta x}=2x+\delta x\,.$$

Finally, I think you will agree that the limit of this value as $\delta x \to 0$ is $2x$. Therefore if we are required to differentiate the function

$$y = f(x) = x^2\,,$$

then the solution is:

$$\frac{dy}{dx}=2x.$$