Differentiation rules – The product rule

The product rule applies when the function we are considering may be expressed as a product of two functions. For example, the function $f(x) = x^2 \cdot e^{-x}$ can be regarded as the product of two functions $g(x) = x^2$ and $h(x) = e^{-x}$. A much simpler case is the function $f(x) = x^2$, where we take $g(x) = x$ to be the first function and $h(x) = x$ as the second function.

So, in general for a product function, we have $f(x) = g(x) \cdot h(x)$. Let's return to first principles and ask the question "What is $f(x+\delta x)$?". Well it's just a case of using $x + \delta x$ as the input variable instead of using x. So $f(x) = g(x) \cdot h(x)$ becomes

$$f(x+\delta x) = g(x+\delta x) \cdot h(x+\delta x).$$

In this case, the fractional expression inside the definition of the derivative, namely $\frac{\delta y}{\delta x}$, is given by

$$\frac{f(x+\delta x) - f(x)}{\delta x} = \frac{g(x+\delta x) \cdot h(x+\delta x) - g(x) \cdot h(x)}{\delta x} \qquad (1)$$

Now, by Taylor's Theorem, we have

$$g(x+\delta x) = g(x) + g'(x) \cdot \delta x + g''(\xi) \cdot \frac{\delta x^2}{2}, \text{ for some } \xi \in (x, x+\delta x) \text{ and}$$

$$h(x+\delta x) = h(x) + h'(x) \cdot \delta x + h''(\psi) \cdot \frac{\delta x^2}{2}, \text{ for some } \psi \in (x, x+\delta x).$$

So, the numerator of the right-hand-side of equation (1) is

$$g(x+\delta x) \cdot h(x+\delta x) - g(x) \cdot h(x)$$

$$= \left\{ g(x) + g'(x).\delta x + g''(\xi).\frac{\delta x^2}{2} \right\} \cdot \left\{ h(x) + h'(x).\delta x + h''(\psi).\frac{\delta x^2}{2} \right\} - g(x).h(x)$$

$$= g(x).h(x) + g(x).h'(x).\delta x + g(x).h''(\psi).\frac{\delta x^2}{2} + g'(x).h(x).\delta x + g'(x).h'(x).\delta x^2$$
$$+ g'(x).h''(\psi).\frac{\delta x^3}{2} + g''(\xi).h(x).\frac{\delta x^2}{2} + g''(\xi).h'(x).\frac{\delta x^3}{2} + g''(\xi).h''(\psi).\frac{\delta x^4}{4} - g(x).h(x)$$

$$= g(x) \cdot h'(x) \cdot \delta x + g'(x) \cdot h(x) \cdot \delta x + O(\delta x^2)$$

Therefore, equation (1) can be written as follows:

$$\frac{f(x+\delta x)-f(x)}{\delta x}=\frac{g(x)\cdot h'(x)\cdot \delta x+g'(x)\cdot h(x)\cdot \delta x+O(\delta x^2)}{\delta x}=g(x)\cdot h'(x)+h(x)\cdot g'(x)+O(\delta x)$$

Therefore, taking the limit as $\delta x \to 0$, we can write the derivative as:

$$f'(x)=\lim_{\delta x \to 0}\{g(x).h'(x)+h(x).g'(x)+O(\delta x)\}=g(x)\cdot h'(x)+h(x)\cdot g'(x).$$

A useful 'saying' that can and should be committed to memory is "1st function multiplied by the derivative of the 2nd plus the 2nd function multiplied by the derivative of the 1st".

Example

Suppose we are asked to differentiate the function $f(x)=x^2\cdot e^{-x}$. Without any knowledge of the product rule, this would be very difficult in the least, if not impossible! However, armed with the product rule, this is quite a simple example to solve.

Now the application of the product rule will get easier and easier the more you apply it to problems, but we've only just learned it, so let's take it easy ...

First, let the first function be $g(x)=x^2$ and the second function be $h(x)=e^{-x}$. Notice that this 'breakdown' of the function $f(x)$ into two component parts will hardly ever be as straighforward as in this example. The next step is to create a table whose first column represents the first function and whose second column relates to the second function. The first and second rows represent the function itself and the derivative respectively. So for this example, the table looks as follows:

	Function 1, $g(x)$	Function 2, $h(x)$
Function	$g(x)=x^2$	$h(x)=e^{-x}$
Derivative	$g'(x)=2x$	$h'(x)=-e^{-x}$

Now, the saying "1st function multiplied by the derivative of the 2nd plus the 2nd function multiplied by the derivative of the 1st" means multiply the leading diagonal terms together and add the product of the off-diagonal terms, giving

$$f'(x)=x^2\cdot\left(-e^{-x}\right)+e^{-x}\cdot 2x=(2-x)\cdot x\cdot e^{-x}$$

Simpler example

At the beginning of this section, we mentioned a very simple example, where the function could be differentiated directly (i.e. without using the product rule). This was the example $f(x) = x^2$. Clearly the derivative here is just $f'(x) = 2x$. Let us use the knowledge of this result to verify the product rule in this case. Let $g(x) = h(x) = x$, so the table becomes as follows:

	Function 1, $g(x)$	Function 2, $h(x)$
Function	$g(x) = x$	$h(x) = x$
Derivative	$g'(x) = 1$	$h'(x) = 1$

In this case, we get $f'(x) = x.1 + x.1 = 2x$, which we have just shown to be correct!