Differentiation rules – The quotient rule

The quotient rule applies when the function we are considering may be expressed as a quotient (fraction) of two functions. For example, the function $f(x) = \frac{e^x}{x^3}$ can be regarded as the ratio of two functions $g(x) = e^x$ and $h(x) = x^3$.

So, in general for a quotient function, we have $f(x) = \frac{g(x)}{h(x)}$. Let's return to first principles and ask the question "What is $f(x + \delta x)$?". Well it's just a case of using $x + \delta x$ as the input variable instead of using x. So $f(x) = \frac{g(x)}{h(x)}$ becomes

$$f(x + \delta x) = \frac{g(x + \delta x)}{h(x + \delta x)}.$$

In this case, the fractional expression inside the definition of the derivative, namely $\frac{\delta y}{\delta x}$, is given by

$$\frac{f(x + \delta x) - f(x)}{\delta x} = \frac{\frac{g(x + \delta x)}{h(x + \delta x)} - \frac{g(x)}{h(x)}}{\delta x} = \frac{g(x + \delta x) \cdot h(x) - g(x) \cdot h(x + \delta x)}{\delta x \cdot h(x) \cdot h(x + \delta x)} \qquad (1)$$

Now, by Taylor's Theorem, we have

$$g(x + \delta x) = g(x) + g'(x) \cdot \delta x + O(\delta x^2) \text{ and } h(x + \delta x) = h(x) + h'(x) \cdot \delta x + O(\delta x^2) \qquad (2)$$

If we examine the numerator of equation (1) above, and substitute the expressions for $g(x + \delta x)$ and $h(x + \delta x)$ from (2), we see that the numerator is

$$g(x + \delta x).h(x) - g(x).h(x + \delta x) = \{g(x) + g'(x).\delta x + O(\delta x^2)\}.h(x) - g(x).\{h(x) + h'(x).\delta x + O(\delta x^2)\}$$

$$= g(x).h(x) + g'(x).h(x).\delta x - g(x).h(x) - g(x).h'(x).\delta x + O(\delta x^2)$$
$$= h(x).g'(x) \cdot \delta x - g(x).h'(x) \cdot \delta x + O(\delta x^2).$$

The denominator is given by

$$\delta x.h(x).h(x + \delta x) = \delta x \cdot h(x) \cdot \{h(x) + h'(x) \cdot \delta x + O(\delta x^2)\} = (h(x))^2 \cdot \delta x + O(\delta x^2)$$

So dividing, we get

$$\frac{f(x + \delta x) - f(x)}{\delta x} = \frac{h(x) \cdot g'(x) \cdot \delta x - g(x) \cdot h'(x) \cdot \delta x + O(\delta x^2)}{(h(x))^2 \cdot \delta x + O(\delta x^2)}$$

$$= \frac{h(x)\cdot g'(x) - g(x)\cdot h'(x) + O(\delta x)}{\left(h(x)\right)^2 + O(\delta x)}$$ (divide top and bottom by δx)

Therefore, taking the limit as $\delta x \to 0$, we can write the derivative as:

$$f'(x) = \lim_{\delta x \to 0} \left\{ \frac{h(x)\cdot g'(x) - g(x)\cdot h'(x) + O(\delta x)}{\left(h(x)\right)^2 + O(\delta x)} \right\} = \frac{h(x)\cdot g'(x) - g(x)\cdot h'(x)}{\left(h(x)\right)^2}$$

So the saying we use to remember how to differentiate quotients is as follows: "bottom, derivative top minus top, derivative bottom all divided by the bottom squared".

Example

Let us consider the example at the beginning of this section, namely the quotient function

$$f(x) = \frac{e^x}{x^3}.$$

First of all, we split this into a 'top function' $g(x) = e^x$ and a 'bottom function' $h(x) = x^3$.

Now, we differentiate each of these functions, to get $g'(x) = e^x$ and $h'(x) = 3x^2$. So the derivative is then

$$f'(x) = \frac{h(x)\cdot g'(x) - g(x)\cdot h'(x)}{\left(h(x)\right)^2} = \frac{x^3\cdot e^x - e^x\cdot 3x^2}{\left(x^3\right)^2} = \frac{e^x\left(x^3 - 3x^2\right)}{x^6} = \frac{e^x\left(x-3\right)}{x^4}.$$