

The limit

Introduction – ‘Freddie the halfway frog’

Let's take a lighthearted look at the idea of the limit. This example makes for a perfect illustration of the concept and so I see no reason not to use it.

Once upon a time, there was a frog called Freddie who lived on a lily in the middle of a pond. One day, a pelican scooped Freddie up in its beak and carried him for a while until he dropped Freddie in the meadow that contained the pond.

Freddie had lots of fun frolicking about in the fields and played for hours before he realised he was tired and needed to make his way back to his lily. So he hopped and hopped all the way back to the edge of his pond. His own lily, stationed right in the middle suddenly seemed very far away from where he was standing and he didn't think he would make it in one leap!

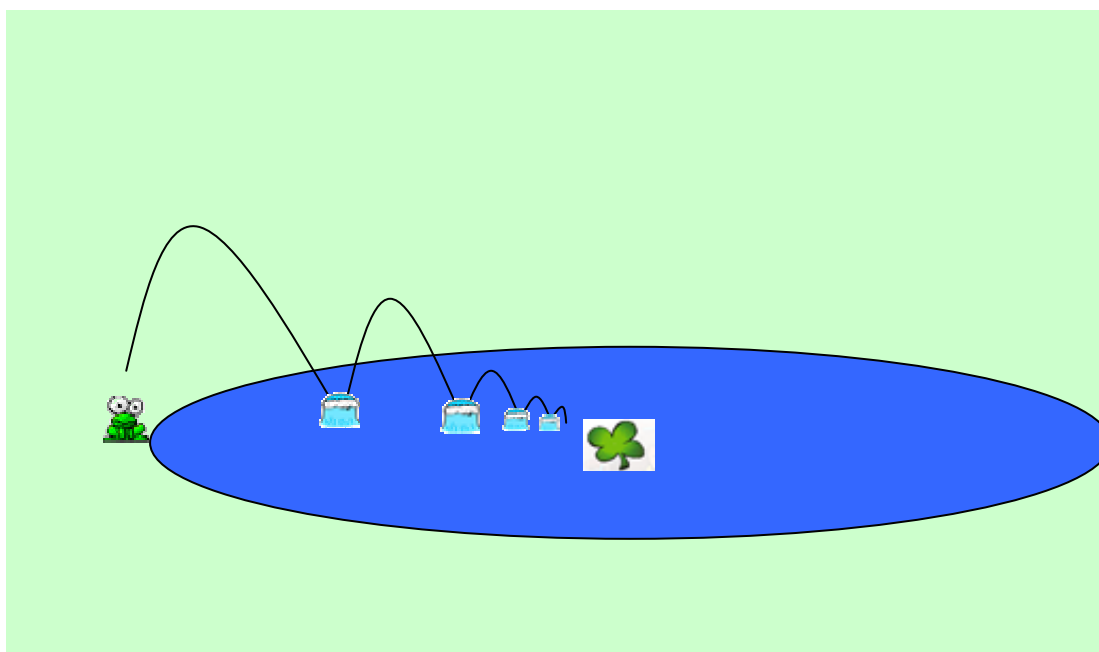
So he jumped as hard as he could from the water's edge. A short while later, there was a big SPLASH ... Freddie had landed in the water exactly halfway from the pond edge and the lily.

Now, in the water, it would take even more effort to jump the remaining distance to his lily, since there was no solid bank to push off from. "I'll jump as hard as I can again!", thought Freddie as he prepared himself for a giant leap. He launched himself out of the water with his powerful back legs and ... SPLASH! ... he ended up halfway between the place where he jumped from and his lily.

This continued, each time Freddie got more and more tired and each jump was half as long as the previous one.

Well there's not much to this story in terms of literacy quality, but if you told it to a small child, the question you'd be asked is "Did Freddie ever make it to his lily?". Of course, the correct answer is "No he didn't", - but why is that?

Well Freddie's first jump wasn't long enough to get to his lily and every attempt after that he only covered half the distance he needed to. So, if you need any more convincing, just imagine any one of his jumps. For that jump, he needed to cover a certain distance, yet he only actually managed half of it.



The next question you might ask is “How close did he get to his lily, then?”. The answer to this question depends of course on which jump you are referring to. If you mean the first jump, then he got half way there. If you mean the second jump, then he got $\frac{3}{4}$ of the way (in total for both jumps), for the third jump, he got $\frac{7}{8}$ of the way.

You should see a pattern emerging here. I’ll lay the total distances out in a table so you can hopefully see it more clearly. In the table, the distances shown are in terms of the proportion of the total distance from the pond-edge to the lily.

Jump number	Distance of that jump	Total distance covered
1	$\frac{1}{2}$	$\frac{1}{2}$
2	$\frac{1}{4}$	$\frac{1}{2} + \frac{1}{4} = \frac{3}{4}$
3	$\frac{1}{8}$	$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$
4	$\frac{1}{16}$	$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{15}{16}$
5	$\frac{1}{32}$	$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} = \frac{31}{32}$
6

Notice that in the ‘total distance covered’ column above, the numerator is always one less than the denominator. Also, the denominators keep on doubling. For jump number 2, the denominator is 4, which is 2^2 . For jump 3, the denominator is 8, which is 2^3 . For jump number 4, the denominator is 16, which is 2^4 .

So, for jump number 10, the denominator would be $2^{10} = 1024$ and the numerator would be one less, which is 1023, so Freddie would have covered

$$\frac{1023}{1024} = 0.999023438$$

of the total distance!

I think you'll agree that the total distance covered after jump number n is given by the formula

$$\frac{2^n - 1}{2^n}$$

Now this value is always less than one (which proves Freddie never makes it!) because the numerator is always less than the denominator. But as n gets larger, the value of the formula changes as shown in the table below:

n	$\frac{2^n - 1}{2^n}$
5	$31/32 = 0.96875$
10	$1023/1024 = 0.999023438$
15	$32767/32768 = 0.999969482$
20	$1048575/1048576 = 0.999999046$
25	0.99999997

The number gets closer and closer to one, but never quite reaches it. If you try this for higher values of n on a handheld calculator (say for $n = 50$), the calculator will probably tell you that the answer is one, but that's only because the calculator doesn't have enough precision to calculate fractions to such a high accuracy. The fact that the numerator is one less than the denominator proves that the answer can never reach 1.

[The more studious reader will notice that the total distance that Freddie jumped at any stage is actually a Geometric Progression. Please look at my page on GPs for more details.]

This is the whole point of the limit concept: that you can get as close as you like to something without ever reaching it. The limit is used a lot in differentiation theory, where you have the idea of two points on a curve approaching one another and getting as close as you want to each other (and if you still aren't satisfied ... bring them even closer!!!! And even closer if you want! And so on.). Whenever two points have some space between them, you can bring them closer together. But, just as importantly in limit theory, they must never touch.

Now I must dash to rescue my dear friend Freddie!