Geometric Progressions

An arithmetic progression is the sum of a set of numbers that are an equal multiple of each other. For example, the sum

$$3+12+48+192+768+3072+12288$$

is a geometric progression (or GP for short). The ‘multiplying factor’ is 4 and the lowest number in the set is 3. Also, the number of numbers in the sequence is 7.

What we wish to do here is develop a formula to give the overall sum, which in the above case is 16383.

First, let the sum be S, then we write the equation for S as follows:

$$S = 3 + 12 + 48 + 192 + 768 + 3072 + 12288$$

Next, we multiply this equation by the ‘multiplying factor’ (in this case 4), to get

$$4\times S = 12 + 48 + 192 + 768 + 3072 + 12288 + 49152$$

Now notice that the coloured numbers in each equation are the same. So let us subtract the first equation from the second – the left-hand-sides give

$$4\times S - S = 3\times S$$

and the right-hand-sides give

$$(12+48+192+768+3072+12288+49152)-(3+12+48+192+768+3072+12288)$$
$$=49152-3$$

because the coloured terms just cancel out through subtraction!

So, we have

$$3\times S = 49152-3 = 49149$$

or

$$S = \frac{49149}{3} = 16383$$

which is the correct answer!

Well if should be possible to see a formula here, if you look hard enough. Let’s look again at the equation

$$3\times S = 49152-3 \qquad (1)$$

Where do these numbers come from? Well, let's think about what we did earlier. First, we wrote the equation for the sum S as

$$S = 3 + 12 + 48 + 192 + 768 + 3072 + 12288$$

Then, we multiplied the equation by the multiplying factor (4) to give

$$4 \times S = 12 + 48 + 192 + 768 + 3072 + 12288 + 49152$$

Then, we subtracted the first equation from the second. So the 3 on the left-hand-side of equation (1) comes from 'the multiplying factor – 1'. Let's call the multiplying factor r (r for ratio).

The number 3 on the right-hand-side is just the first term of the sum's sequence and the number 49152 came from multiplying the highest term of the sequence by the multiplying factor.

If the first term of the sequence is called a, then the second term is $a \times r = ar$, the third term is $a \times r \times r = ar^2$, the fourth term is ar^3, and so on. In particular, if the sum has n terms (7 in our case), then the n^{th} term is ar^{n-1}. Using this notation, the original sum could be written as follows:

$$S = a + ar + ar^2 + ar^3 + \ldots + ar^{n-2} + ar^{n-1}$$

The second equation comes from multiplying throughout by r, giving

$$rS = ar + ar^2 + ar^3 + \ldots + ar^{n-1} + ar^n$$

Subtracting as before, we get

$$(r-1)S = ar^n - a = a\left(r^n - 1\right)$$

Dividing through by $(r-1)$, we get the final form:

$$S = \frac{a\left(r^n - 1\right)}{r-1}$$

Let's try our example again! Here it is:

$$S = 3 + 12 + 48 + 192 + 768 + 3072 + 12288$$

The first term of the sum is 3, so set $a = 3$. The multiplying factor is 4, so set $r = 4$. The number of terms is 7, so $n = 7$.

Now try the formula …

$$S = \frac{3 \cdot \left(4^7 - 1\right)}{4-1} = \frac{3 \cdot (16384 - 1)}{3} = 16383.$$

Try some more for yourself!